Lecture 18

Deflection of light

Objectives:

- *Deflection of light*

Reading: *Schutz, 10 & 11; Hobson 9 & 10; Rindler 11.*

18.1 Circular photon orbits

Show gravitational lensing pictures.

$V = h^2/2r^2(1 - 2\mu/r)$.

Circular orbits: $r = r_C$ such that $V'(r_C) = 0$, or

$$-\frac{1}{r_C^3} + \frac{3\mu}{r_C^4} = 0,$$

i.e.

$$r_C = \frac{3GM}{c^2}.$$

3× the Newtonian result $r_C = GM/c^2$, problem sheet 1.

18.2 Deflection of light by the Sun

Orbits with $r \gg GM/c^2$ suffer a small deflection which is experimentally measurable.

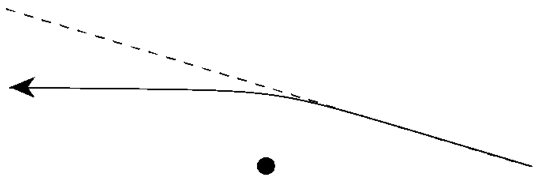

Figure: Deflection of light by a mass. Black circle shows the event horizon, so the deflection in this case is large.

To calculate light deflection, we need an equation relating r and ϕ without the affine parameter, λ. Can obtain this by noting:

$$\dot{r} = \frac{dr}{d\lambda} = \frac{dr}{d\phi}\frac{d\phi}{d\lambda} = \dot{\phi}\frac{dr}{d\phi} = \frac{h}{r^2}\frac{dr}{d\phi}.$$

Then the energy equation becomes

$$\frac{h^2}{r^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{h^2}{r^2}\left(1 - \frac{2\mu}{r}\right) = c^2k^2.$$

Making the substitution $r = 1/u$ (also used for Newtonian orbits):

$$u^4\left(-\frac{1}{u^2}\frac{du}{d\phi}\right)^2 + u^2(1 - 2\mu u) = \frac{c^2k^2}{h^2},$$

and so

$$\left(\frac{du}{d\phi}\right)^2 + u^2 - 2\mu u^3 = \frac{c^2k^2}{h^2}.$$

Finally, differentiating with respect to ϕ and dividing by $2du/d\phi$:

$$\frac{d^2u}{d\phi^2} + u = 3\mu u^2.$$

For large radii, $r \gg \mu$, $u \ll \mu^{-1}$, the RHS can be neglected and we have the SHM equation, thus:

$$u = a\sin\phi + b\cos\phi,$$

where a and b are constants, or, without loss of generality, simply

$$u = a\sin\phi,$$

or $r\sin\phi = 1/a = r_0$, a constant. This is the equation of a straight line with impact parameter r_0. As $r \to \infty$, $u \to 0$ gives $\phi = 0$ or π.

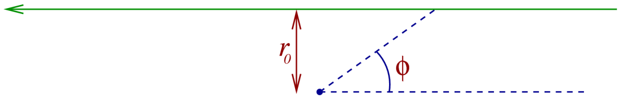

Figure: Straight from $\phi = 0$ to $\phi = \pi$ with polar equation $r \sin\phi = a^{-1}$.

Now look for a better approximation $u = u_0 + u'$ where $|u'| \ll u_0 = a\sin\phi$. Then

$$\frac{d^2u'}{d\phi^2} + u' = 3\mu u^2 \approx 3\mu u_0^2 = 3\mu a^2 \sin^2\phi = \frac{3\mu a^2}{2}(1 - \cos 2\phi),$$

neglecting small terms on the right. Particular integral is

$$u' = \frac{3\mu a^2}{2}\left(1 + \frac{1}{3}\cos 2\phi\right),$$

so a better solution is

$$u = a\sin\phi + \frac{3\mu a^2}{2}\left(1 + \frac{1}{3}\cos 2\phi\right).$$

Now $r \to \infty \implies u = 0 \implies$

$$\sin\phi = -\frac{3\mu a}{2}\left(1 + \frac{1}{3}\cos 2\phi\right) \approx -2\mu a,$$

since $\cos 2\phi \approx 1$ for $\phi = 0,\ \pi$. Therefore

$$\phi \approx -2\mu a, \quad \text{or} \quad \pi + 2\mu a.$$

Thus light is deflected by

$$\boxed{\Delta\phi = 4\mu a = \frac{4GM}{c^2 r_0}.}$$

This is 2× the Newtonian result (pure SR predicts zero).

For light grazing the Sun

$$\Delta\phi = \frac{4GM_\odot}{c^2 R_\odot} = \frac{4 \times 6.67 \times 10^{-11} \times 2 \times 10^{30}}{(3 \times 10^8)^2 \times 7 \times 10^8} = 8.47 \times 10^{-6}\,\text{rads} = 1.75\,\text{arcsec}.$$

Confirmed from observations of radio sources to 2 parts in 10^4 Deflection of light now an important tool in astronomy, "gravitational lensing".

See Shapiro et al in reading.

Famously tested by British astrophysicist Eddington in 1919 using observations of stars near the Sun during a total eclipse. Made Einstein famous. Eddington the source of the well-known quote "Interviewer: Professor Eddington, is it true that only three people understand Einstein's theory? Eddington: Who is the third?"